Modeling the Thrust Phase

Version 0.6
Dean Wheeler

I. Introduction

Water rockets can achieve upward velocities on the order of 70m/s (157 mph). For this to happen, a significant amount of the energy stored in the pressurized gas is converted into kinetic energy of the rocket. This converstion is a very rapid process, since the thrust phase generally takes about 1/10 second. The use of a launch tube and the proper amount of water in the rocket serve to increase the efficiency of this conversion.

This multi-part document describes my mathematical approach to modeling the thrust phase of a conventional water-rocket. By conventional I mean a vertically flying rocket which uses a pressurized inert gas, principally stored inside the rocket, as its energy source. Although I use the general term *water rocket*, the analysis also covers the case of a pressurized rocket containing no water.

I assume that the reader is familiar with calculus and basic classical physics. Some knowledge of fluid dynamics is also helpful. I expect that this document will be mostly of interest to hard-core types who write and use water-rocket simulations. I have incorporated much of the material here into a java applet simulator available at my website:
http://www.cchem.berkeley.edu/~jsngrp/dean/benchtop

In general the governing equations do not permit an analytical (exact) solution to rocket velocity as a function of time. Rather, we must get a numerical computer solution to a set of equations. The rocket velocity and altitude at the point of 'burnout' (when the pressure energy has been exhausted) can be used subsequently to estimate rocket apogee height and time using analytical equations I describe separately on my website.

The thrust portion of flight for a water rocket is composed of up to three phases, each identified by the thing that is being expelled from the rocket at the time: (1) *launch tube*, (2) *water*, and (3) *gas*. I will discuss these phases in turn after presenting some preliminary mathematics they have in common.

II.Preliminary Mathematics

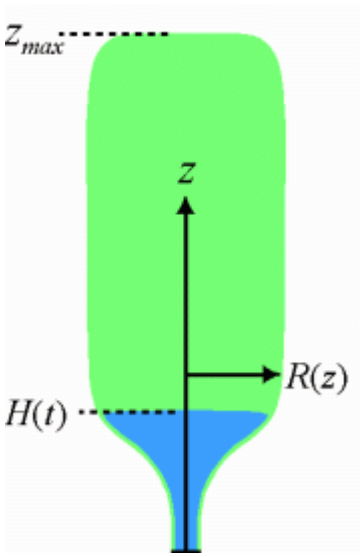

Fig. 1 Bottle-based reference frame.

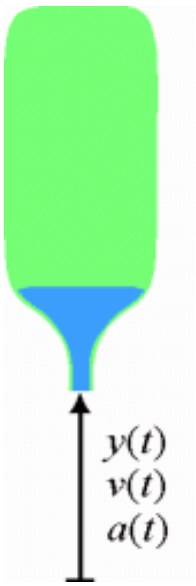

Fig. 2 Ground-based reference frame.

Reference Frames

First I will establish the reference frames or coordinate systems to be used. Fig. 1 shows a bottle-based reference frame where *z* describes the axial distance from the bottom or outlet of the bottle. Whenever I use the word *axial* in this document, I mean in the *z* direction.

$R(z)$ gives the radius of the bottle as a function of *z* and $H(t)$ is the height of the water/gas interface as a function of time.

A second reference frame is shown in Fig. 2 and takes the launch pad as its origin, y = 0. $y(t)$, $v(t)$, and $a(t)$ are respectively the altitude, velocity, and acceleration of the bottle as functions of time. I will assume vertical (or nearly vertical) flight of the rocket during thrust. To adapt the equations for nonvertical flight requires a few substitutions such as replacing the gravitational constant *g* with $g\cos\vartheta$ where ϑ is the angle the reference frame of Fig. 2 takes with respect to vertical.

Bottle Shape

The bottle-based reference frame (Fig. 1) will be the one in which fluid flow in the bottle will be determined. The function $R(z)$, required as input, plays an important role in those calculations, and several other functions will be derived from it. First among these is the local cross-sectional bottle area:

$$A(z)=\pi R^2(z)\,.$$

The shape of the bottle can be coarse-grained if desired to simplify the analysis. Fig. 3 gives some examples where the radius function *R*(*z*) is composed of successive linear segments. The rightmost shape obviously affords the easiest analysis, but sacrifices some accuracy.

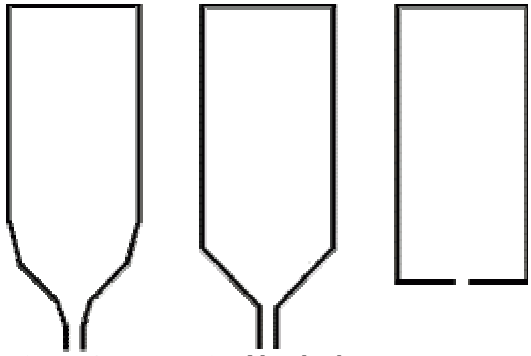

Fig. 3 Coarse-grained bottle shapes.

The nozzle region is that part of the bottle where R(z) becomes small, in the vicinity of the outlet. I use the term 'throat' to refer to the most narrow part of the nozzle. In defining the shape of the bottle with *R*(*z*), it is more important that it be accurate in the nozzle region rather than the 'bulk' or 'tank' region (where *R* is large) if a tradeoff must be made.

Unidirectional Flow

One of the most important simplifying assumptions I make is that the fluid flow within the bottle is unidirectional or mostly 1-dimensional. That is, non-axial components of the fluid velocity are small compared to the axial (*z*-direction) component and can be neglected. In real life there will be non-axial components of flow, but in many cases their explicit inclusion in the equations would not change the model predictions substantially. One consequence of this assumption is that it eliminates radial variations in fluid pressure, temperature, and density. Note that the term *fluid* in this document means either the gas (usually air) or the water. For the unidirectional flow assumption to be reliable, the taper of the bottle must be gradual rather than abrupt in the nozzle region. This is not too bad of an assumption for soda pop bottles.

The function *u*(*z*,*t*) designates the axial fluid velocity relative to the bottle and $\rho(z,t)$ is the fluid density. For $z<H(t)$ (below the interface) the density is the constant value for water, $\rho_w \approx 998\,\mathrm{kg/m^3}$. For $z>H(t)$ (above the interface) the density is that of the compressed gas, and will vary with time.

A few other useful variables are

$$u_{out}(t)=u(0,t)$$
$$\rho_{out}(t)=\rho(0,t)$$
$$A_{out}=A(0)$$

where the subscript *out* indicates a property at $z = 0$ or the bottle outlet. Note that fluid velocity u and u_{out} will always take negative values indicating downward flow out of the bottle.

Mass and Volume

We must know the mass of the rocket at all times. The bottle mass (including fins, nose cone, etc.) is a fixed value, m_b. The mass of the fluid is determined by integrating density over the volume of the bottle. The total rocket mass is thus

$$\begin{aligned} m_{tot} &= m_b + \int \rho \, dV \\ &= m_b + \int_0^{z\,max} \rho \, A \, dz \end{aligned} \tag{1}$$

When the limits are not specified on an integral over dV, it is implied to be over the entire bottle volume, V_b. Note that the nomimal volume of soda bottles (2 liters, for instance) is not necessarily the real volume of the bottle since there is usually additional gas space above the soda pop.

Also, because bottles constructed of PET and other plastics stretch somewhat upon pressurization, I usually assume V_b for the pressurized bottle to be 3% greater than the bottle volume at ambient pressure. While it is true that this augmented volume will disappear as the bottle depressurizes, the important thing is to get a correct accounting for the energy stored in the compressed gas initially occupying that augmented volume. The relaxation of the stretched plastic walls is adding yet more energy to the system, but I believe that it is minor in comparison to the compressed gas energy.

For use later I also define a function giving the amount of water in the bottle based on the height of the gas-water interface at any given time:

$$V_w(H) = \int_0^H A \, dz \, . \tag{2}$$

V_{w0} and H_0 will indicate the initial values for the respective quantities.

Equation of Bottle Motion

A typical starting point for describing classical motion is Newton's Second Law:

$$m \frac{dv}{dt} = m\,a = F$$

Unfortunately, this equation only works for a closed system, or one that does not exchange mass with its surroundings. In order to treat the water rocket, an open system, we need a more general equation of motion:

$$\underbrace{\frac{d}{dt}\left(m_b v + \int \rho(u+v)\,dV\right)}_{\text{rate of change in rocket momentum}} = \underbrace{(P_{out}-P_{atm})A_{out} + F_{drag} - m_{tot}g}_{\text{external forces}} + \underbrace{\mathring{m}(u_{out}+v)}_{\text{momentum flow thru outlet}} \qquad (3)$$

where $(u+v)$ is the velocity of the fluid relative to the ground, $(P_{out}-P_{atm})$ is the pressure difference between the exiting fluid and the atmosphere, the external drag force is

$$F_{drag} = -\tfrac{1}{2} C_D \rho_{atm} A_b v\,|v| \qquad (4)$$

and the gravitational force on the rocket is $-m_{tot}g$. $\mathring{m}$ in eq. (3) is the rate of change in mass of the rocket, and can be determined by looking at mass flow through the outlet:

$$\mathring{m} = \frac{dm_{tot}}{dt} = \rho_{out} u_{out} A_{out}\,. \qquad (5)$$

Note that $\mathring{m}$ takes a negative value. The left side of eq. (3) can be shown to be

$$\frac{d}{dt}\left(m_b v + \int \rho(u+v)\,dV\right) = m_{tot}a + \mathring{m}v + \frac{d}{dt}\left(\int \rho u\,dV\right)\,. \qquad (6)$$

Next we make the substitutions

$$F_{int} = -\frac{d}{dt}\left(\int \rho u\,dV\right) \qquad (7)$$

$$\text{and} \quad F_{thr} = (P_{out}-P_{atm})A_{out} + \mathring{m}u_{out} \qquad (8)$$

in eq. (3) and with a bit of rearranging get an expression for the acceleration of the bottle:

$$a = \frac{F_{thr}+F_{int}+F_{drag}}{m_{tot}} - g \qquad (9)$$

The F_{int} Term

As one might expect, F_{thr} of eq. (8) is the traditional thrust force for rockets and F_{drag} the external drag force, but what is F_{int}? It is a reaction force due to internal acceleration of mass in the rocket.

There are many suitable analogies that explain why this term is necessary. Here is one: consider a cannon firing a ball. When the powder charge ignites, the ball begins to accelerate down the barrel. If one were to calculate the thrust term, F_{thr}, for the cannon as defined above, it would be zero until the instant the ball exits the barrel. If there were no term F_{int} then eq. (9) would predict that the cannon would feel no force and would not recoil until the instant the ball exits the barrel. Is this right? No, because the cannon recoils continuously as the ball accelerates down the barrel. To correctly model this requires the inclusion of a term such as F_{int}. This term does *not* change the total amount of recoil or reaction force acting on the cannon, but merely redistributes it properly in time.

For some rockets F_{int} can safely be neglected, either because the movement of mass within the rocket is relatively constant in time or because we only care about the total impulse. In the case of water rockets, however, the acceleration of the rocket couples to the water flow equation and so F_{int} should not be neglected during the water-impulse phase.

Getting a Solution

Eq. (9) gives the bottle acceleration in time. Additional equations discussed below are needed to generate the pressure and flow terms that go into eqs. (7) and (8). To relate the acceleration to the bottle velocity v and altitude y requires the following two differential equations be coupled to the rest

$$\frac{dv}{dt} = a \;;\; \frac{dy}{dt} = v \quad . \tag{10}$$

All the differential equations must be simultaneously integrated over time in order to get a solution. It is a relatively routine matter to generate a solution by computer for these kinds of equations using a finite-differencing scheme (for an example, see Appendix C).

Ideal Gas Law

This is the simplest *equation of state* relating the various properties of a gas such as pressure, temperature, and density. It is generally valid around ambient pressure and temperature. The pressures typical of soda-bottle rockets will not cause significant deviations from the ideal gas law. It is

$$P = \rho R_M T \tag{11}$$

where R_M is the ideal gas constant (8.3145 J/mol-K) divided by the molar mass of the gas (0.028964 kg/mol for air). P and T are absolute pressure and temperature, respectively. Note that absolute pressure is gauge pressure plus the ambient or atmospheric pressure P_{atm}. In fact, for consistency all pressure variables used in this document will be absolute pressures.

Adiabatic Expansion

The expansion of the compressed gas in the rocket is its principle source of energy. I will assume that the compressed gas expands adiabatically. This means that it expands in such a short time frame that heat cannot flow fast enough from the outside world into the gas to keep it at a constant temperature. As a consequence the gas cools as it expands.

This is probably a good assumption given that the typical thrust phase lasts around one-tenth of a second. During this time outside heat will penetrate only about a millimeter from the walls into the gas in the bottle, due to the insulating properties of the gas. Also consistent with the adiabatic assumption is the oft observed formation of fog inside the bottle during launch, due to cooling of the internal gas.

Without delving into how they are derived, I will simply state the adiabatic gas relations

$$\begin{aligned} P\,\rho^{-\gamma} &= \text{constant} \\ T\,\rho^{1-\gamma} &= \text{constant} \end{aligned} \qquad (12)$$

where $\gamma = C_P/C_V$ is the ratio of specific heats. For air the ratio is very close to 7/5 or 1.4. The above equations basically allow us to connect the conditions of the gas (*P, T,* or ρ) at one time or location with the conditions of the gas at another time or location.

Now we are in a position to examine the three thrust phases in greater detail, beginning with the launch tube (that is, if you are still with me at this point).

III.Launch-Tube Phase

The launch tube acts as an internal piston, and exerts a force on the rocket equal and opposite to a force the rocket exerts on the tube. The launch tube phase lasts so long as *y*, the height of the rocket from its initial position, is less than L_T, the length of the tube.

During the launch-tube phase I assume that a negligible amount of fluid escapes the rocket, that is the launch tube forms a close but not tight fit inside nozzle. The cross-sectional area of the tube based on its outside diameter is denoted by A_{TO}, while the area based on the inside diameter (the hole in the tube) is denoted by A_{TI}.

Equation of Motion

To calculate the acceleration of the rocket we start with the appropriately simplified version of eq. (9):

$$a = \frac{F_{thr} + F_{drag}}{m_{tot}} - g \qquad (13)$$

While there is some rearrangement of fluid within the bottle as the rocket moves up the launch tube, F_{int} of eq. (7) is generally negligible.

In order to get the thrust force of eq. (8), we need to consider the movement of gas within the system. Assuming the launch tube has a moderately sized opening, there will be rapid fluid communication between the launcher apparatus volume and the bottle volume. This means they will share a common pressure, which will decrease in response to an increase in shared volume. The volume change produces a density change according to

$$\frac{\rho}{\rho_0} = \frac{V_{init}}{V_{init} + y\,A_{TO}} \quad ; \qquad (14)$$

$$V_{init} = V_l + V_b - V_{w0} - L_T(A_{TO} - A_{TI}) \qquad (15)$$

The left side of eq. (14) is the the density of gas in the system over its initial value. V_{init} is the initial gas volume of the combined system and is broken into its constituent parts in eq. (15). V_l is the launcher gas volume not including the launch tube itself, V_b is the empty bottle volume, and V_{w0} is the initial amount of water added to the bottle given by eq. (2). The final terms in eq. (15) account for the volume inside the rocket initially occupied by the launch tube.

In response to the decrease in gas density, the pressure and temperature decrease adiabatically. The applicable forms of eqs. (12) are

$$\frac{P}{P_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma} \quad ; \quad \frac{T}{T_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma - 1} \qquad (16)$$

where the subscript 0 again indicates initial values in the bottle.

The force on the rocket comes simply by the difference between the internal and external pressure:

$$F_{thr} = (P - P_{atm})\,A_{TO} \qquad (17)$$

The Solution

To summarize for the launch tube phase, equations (10) are the differential equations which must be solved along with auxiliary equations (13) through (17). A solution explicit in time can be obtained only by numerical integration.

However, there is an analytical solution for velocity as a function of *position*:

$$v(y)=\left\{\frac{2P_0V_{init}}{m_{tot}(1-\gamma)}\left[\left(\frac{y\,A_{TO}}{V_{init}}+1\right)^{1-\gamma}-1\right]-\left(\frac{P_{atm}A_{TO}}{m_{tot}}+g\right)2y\right\}^{1/2} \qquad (18)$$

Eq. (18) is obtained by doing a mechanical energy balance on the system and assuming that F_{drag} is negligible:

$$\frac{1}{2}m_{tot}v^2=\int_0^y F_{thr}\,dy-m_{tot}g\,y \qquad (19)$$

Values such as gas pressure and density recorded at the conclusion of the launch-tube phase ($y = L_T$) will be given a subscript 1.

Side note: bottles will fail if the internal pressure causes too much stress on the walls. For most rockets that contain water, the maximum pressure during launch will be in the vicinity of the bottle outlet, at the moment just before the bottle has cleared the launch tube. The pressure applied there will be approximately $P_{max}\approx P_1+(a_1+g)\rho_w H$ where P_1 and a_1 are evaluated at $y = L_T$.

IV. Water-Impulse Phase

If water is contained in the rocket it will then be expelled due to the force of the high-pressure gas. A few additional equations need to be added to our pantheon to describe the flow of water through and out of the bottle.

Mass Continuity

Mass continuity, also known as mass conservation, is important in almost all fluid flow problems. Eq. (5) alludes to this principle. The general version that applies to our unidirectional flow is

$$\frac{d}{dt}(\rho A)+\frac{d}{dz}(\rho u A)=0 \qquad (20)$$

This equation basically states that the rate of change in mass contained in a thin slice of fluid in the bottle is related to the difference in flow into and out of that slice.

When describing flow of the water, the first term in eq. (20) vanishes since we can treat water as incompressible (constant density ρ_w). The second term reduces to

$$u\,A = u_{out}\,A_{out} = \text{function of time only} \tag{21}$$

Since $A(z)$ is known beforehand, this means that we can express the water velocity at any z value in terms of one variable, the exit velocity u_{out}.

Water Motion from the Bernoulli Equation

The celebrated Bernoulli Equation is a statement of conservation of energy in a frictionless irrotational flow of fluid. (However, as discussed in Appendix A it can be modified in an ad hoc manner to account for frictional losses.) The form that many textbooks present is a Bernoulli Equation for *stationary* or *steady state* flow. However, the flow in water rockets is highly transient—the water starts out in a quiescent state and is rapidly accelerated. The transient Bernoulli equation for water flow in the bottle can be stated as:

$$\int \left(\frac{\partial u}{\partial t}\right) \cdot dz + \Delta\left[\frac{u^2}{2} + \frac{P}{\rho_w} + (a+g)\,z\right] = 0 \tag{22}$$

where the integral and the difference (Δ) are together taken between any two locations (z values) in the water. Eq. (22) is a differential equation. In the steady-state case the differential term on the left would vanish and an algebraic equation would be left behind. Also unusual in the present case is the presence of a in the $(a + g)\,z$ term, which accounts for the fact that the bottle reference frame to which u and z are referenced is accelerating in the vertical direction.

The most convenient two locations or limits to use in eq. (22) are the outlet ($z = 0$) and the gas/water interface ($z = H(t)$). The resulting equation is

$$\int_0^H \left(\frac{\partial u}{\partial t}\right) \cdot dz + \frac{u_H^2 - u_{out}^2}{2} + \frac{P_H - P_{out}}{\rho_w} + (a+g)\,H = 0 \tag{23}$$

To get eq. (23) into shape we note the following:
(1) Since the water is essentially incompressible, the outlet pressure must be in mechanical equilibrium with the ambient pressure, hence $P_{out} = P_{atm}$.
(2) The water pressure at the interface will be the same as the gas pressure, which depends adiabatically on the density of gas, which in turn depends on the height of the interface. This gives us a relation similar to eqs (14) and (16):

$$P_H = P_1\left(\frac{\rho}{\rho_1}\right)^{\gamma} = P_1\left(\frac{V_b - V_{w0}}{V_b - V_w(H)}\right)^{\gamma} \tag{24}$$

where P_1 and ρ_1 are respectively the gas pressure and density at the beginning of the water-impulse phase.
(3) Mass continuity, eq. (21), allows water velocity u at any location to be related to u_{out}.

The above relationships allow eq. (23) to be transformed into

$$B(H)\frac{du_{out}}{dt} + C(H)u_{out}^2 + D(H)\frac{P_1}{\rho_w} - \frac{P_{atm}}{\rho_w} + (a+g)H = 0 \tag{25}$$

with newly defined functions of H given as

$$\begin{aligned} B(H) &= \int_0^H \frac{A_{out}}{A(z)}\,dz \\ C(H) &= \frac{1}{2}\left[\left(\frac{A_{out}}{A(H)}\right)^2 - 1\right] \\ D(H) &= \left(\frac{V_b - V_{w0}}{V_b - V_w(H)}\right)^{\gamma} \end{aligned} \tag{26}$$

The value of H is determined on the fly by a coupled differential equation that results from the fact that the interface moves with the same velocity as the adjacent fluid:

$$\frac{dH}{dt} = u_H = \frac{A_{out}\,u_{out}}{A(H)} \tag{27}$$

Equation of Motion

To calculate the acceleration of the rocket we start with the full version of eq. (9):

$$a = \frac{F_{thr} + F_{int} + F_{drag}}{m_{tot}} - g \tag{9}$$

Here is where the internal fluid acceleration term (eq. (7)) finally gets used:

$$\begin{aligned} F_{int} &= -\frac{d}{dt}\int_0^H \rho_w u\,A\,dz \\ &= -\frac{d}{dt}\left(\mathring{m}H\right) \\ &= -\rho_w A_{out}\left(H\frac{du_{out}}{dt} + \frac{A_{out}}{A(H)}u_{out}^2\right) \end{aligned} \tag{28}$$

As for the thrust force, eq. (8), we get the expected relationship

$$F_{thr} = \rho_w A_{out} u_{out}^2 . \qquad (29)$$

We must also keep in mind that the mass of the rocket in eq. (9) will be changing in time according to the interface height. Including both the mass of gas and of water in the total mass, we get $m_{tot} = m_b + \rho_1 (V_b - V_{w0}) + \rho_w V_w(H)$.

Transition from Launch Tube to Water Impulse

It is interesting to compare the computed acceleration of the rocket at the last moment of the launch-tube phase to that at the first moment of the water-impulse phase. Neglecting the deceleration due to air drag, we get:

launch tube: $$a_{1-} = -g + \frac{(P_1 - P_{atm}) A_{TO}}{m_{tot}}$$

water impulse: $$a_{1+} = -g + \frac{(P_1 - P_{atm}) A_{out}}{m_{tot}} \left(\frac{B(H_0)}{H_0} - \frac{\rho_w H_0 A_{out}}{m_{tot}} \right)^{-1} \qquad (30)$$

According to the above equations there is a rapid jump in acceleration as the rocket clears the launch tube. Most of this is due to the reaction force of setting in motion the water relative to the bottle. Nevertheless, the water-impulse phase is less efficient than the launch-tube phase because it requires more energy expense (in terms of gas expansion) for the same amount of thrust.

What is the optimal amount of water to put in the rocket? This must be solved numerically, by varying the amount of water and using a full model to generate apogee height. The water serves as a reaction mass and can increase thrust efficiency over a gas-only rocket. However, the water occupies bottle volume that could otherwise be occupied by energy-containing pressurized gas. There is a trade-off between thrust efficiency and energy storage, resulting in an optimal proportion of water given fixed total volume and other rocket parameters. The optimal proportion of water seems to be largely independent of rocket pressure, and is typically around ¼ for 2-liter soda bottles.

The Solution

Following the initial acceleration given by eq. (30) above, the acceleration during water impulse must be computed by eq. (9) from knowledge of du_{out}/dt, u_{out}, and H. These are obtained by integrating eqs. (25) and (27). When eqs. (10) are added into the mix that makes four ordinary differential equations that must be simultaneously integrated by computer for the

water-impulse phase. The conclusion of this phase is signaled by the condition $H \leq 0$. Values such as gas pressure and density recorded at that time will be given a subscript 2.

In real life, the water-gas interface will become unstable near the conclusion of the water-impulse phase, probably when *H* becomes less than the nozzle diameter. High-speed gas will break through the interface and for perhaps a few milliseconds the fluid exiting the nozzle will be a mixture of water and gas. Some have wondered if this mixed flow will have a synergistic (increased) thrust effect. This is not known. Without further information I choose to ignore it, assuming instead a clean transition between water and gas exiting the nozzle.

V.Gas-Impulse Phase

The governing equations for compressible high-speed gas flows are generally more complex than for incompressible fluids such as water. For instance, in the equation of continuity, eq. (20), we must retain all the terms since ρ can depend on both *z* and *t*. However a major simplification can be made for our rockets. The simplification relies on the 'bulk' or 'tank' region of the bottle being large compared to the nozzle region. If this is the case then two things happen:

(1)In the tank region, velocities and spatial variations in gas density are relatively small and change relatively slowly in time. This is known as a stagnation condition. However, the density still decreases fast enough for adiabatic expansion to continue to hold.

(2)High-speed flow in the nozzle region is quasi-steady state, that is, the gas there responds essentially instantaneously to conditions in the tank region. All the spatial variations in gas density are concentrated in this region.

These observations suggest that we formally divide the bottle into the two regions. Gas properties in the tank region will be given a subscript *t*. The two important locations in the nozzle are the throat or narrowest part (subscript *) and the outlet (subscript *out*).

Fig. 4 shows the two types of nozzles. The one on the left is a converging-diverging or Delaval type, hereafter abbreviated as *CD*. The one on the right is a converging-only type, hereafter abbreviated as *C*, and is the default nozzle on soda bottles. For the *C* nozzle the throat and the outlet coincide.

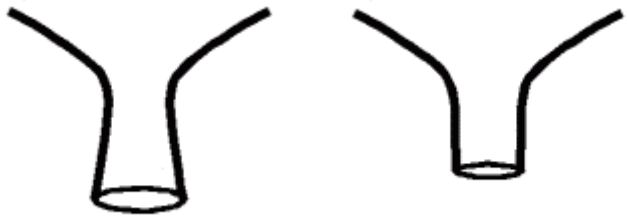

Fig. 4 Nozzles

The mathematics covering the *CD* nozzle is more complex than for the C nozzle. Because the *C* nozzle is the one primarily used on water-rockets, I will emphasize it here.

Mass Continuity

In the tank region, the density depends on the amount of mass which has exited through the nozzle. This is expressed as a differential equation:

$$\frac{d\rho_t}{dt} = \frac{\mathring{m}}{V_b} \tag{31}$$

where ρ_t takes as its initial condition the value ρ_2. Because $\mathring{m}$ is negative, ρ_t will decrease in time.

In the nozzle region, the quasi-steady-state condition is expressed as

$$\rho_* u_* A_* = \rho_{out} u_{out} A_{out} = \mathring{m} \tag{32}$$

where the first equality comes from continuity eq. (20) assuming the first term to be negligible compared to the second; and the second equality comes from eq. (5).

It is convenient to describe the velocity in terms of the local Mach number *M,* defined according to the local speed of sound *c*:

$$\begin{aligned} M &= |u|/c \\ c &= \sqrt{\gamma R_M T} \end{aligned} \tag{33}$$

In the stagnant tank region, *M* is effectively zero.

Nozzle Flow Regimes

There are two different flow regimes in the nozzle. In order of decreasing tank pressure they are

I. Choking flow.
Choking flow means that $M = 1$ exactly in the narrowest part of the nozzle, A_*. This throat condition establishes the maximum $\mathring{m}$ that any nozzle can sustain. This means that a *CD* nozzle and a *C* nozzle each attached to the same tank and having the same A_* will expel mass at the same rate. In general the outlet pressure will not equal ambient and so the pressure part of the thrust force, eq. (8), will be nonzero. For the *C* nozzle choking flow will conclude when the throat pressure P_* drops below ambient pressure. The pressure in the tank at that moment is given as

$$P_{trans} = P_{atm}\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}} \tag{34}$$

The *CD* nozzle is able to accelerate gas to supersonic speeds in its diverging section. During most of the choking flow regime a shock wave will lie just ouside the nozzle exit, where the gas abruptly changes from supersonic to subsonic speeds. However, before the conclusion of the choking flow regime, the shock will move into the diverging section. The presence of the shock inside the *CD* nozzle does not alter the flow rate, but does alter the amount of thrust generated.

II. Purely subsonic flow.
Unlike in the choking flow regime, in purely subsonic flow $\mathring{m}$ will be sensitive to the ratio between inlet and outlet pressures. The outlet pressure is constrained to be ambient pressure. It turns out that the math covering this regime is more difficult than for choking flow. However, it is not necessary to make completely accurate calculations here because this regime comes at the very end of the gas-impulse phase and relatively little energy is left in the bottle to produce thrust.

Other Nozzle Relations

The following relation will give the gas density at any location in the nozzle (given by *M*) by relating it to the stagnant tank density ρ_t.

$$\frac{\rho}{\rho_t} = \left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{1}{1-\gamma}} \tag{35}$$

We can combine eq. (35) with the adiabatic relations to get *P* or *T* at any point in the nozzle defined by its Mach number:

$$\frac{P}{P_t} = \left(\frac{\rho}{\rho_t}\right)^{\gamma} \quad ; \quad \frac{P_t}{P_2} = \left(\frac{\rho_t}{\rho_2}\right)^{\gamma}$$
$$\frac{T}{T_t} = \left(\frac{\rho}{\rho_t}\right)^{\gamma-1} \quad ; \quad \frac{T_t}{T_2} = \left(\frac{\rho_t}{\rho_2}\right)^{\gamma-1} \tag{36}$$

For instance, by setting $M = 1$ we can obtain the conditions in the throat during choking flow. In addition, a simple relation for gas throat velocity under choking flow is

$$|u_*| = c_* = \sqrt{\gamma R_M T_*} = \sqrt{\frac{2\gamma}{\gamma+1} R_M T_t} \tag{37}$$

On the other hand, if the nozzle is operating in regime II, M must be determined at the exit by letting $P_{out} = P_{atm}$. In either case, once we know the Mach number at one location (area A_1) it can be computed for another location (area A_2) in the nozzle by

$$\frac{A_1}{A_2}=\frac{M_2}{M_1}\left[\frac{2+(\gamma-1)M_2^2}{2+(\gamma-1)M_1^2}\right]^{\frac{\gamma+1}{2(1-\gamma)}} \tag{38}$$

Notice that eq. (38) is implicit in M_1 and M_2. In fact, there can be more than one solution for the same area ratio—one solution corresponding to subsonic flow and the other for supersonic flow.

Tank Blowdown Solution

Using a combination of the above relations it is possible to solve differential eq. (31) to get the gas density in the tank as a function of time. This, in turn, can be converted to tank pressure through eq. (36). The solution for regime I (choking flow) is

$$P_t=P_2\left(1+\frac{t}{\tau}\right)^{\frac{2\gamma}{1-\gamma}} \tag{39}$$

where P_2 is the tank pressure at the start of blowdown. The nozzle time constant is given by

$$\tau=\frac{V_b}{A_*\ c_2}\left(\frac{2}{\gamma-1}\right)\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{40}$$

with $c_2=\sqrt{\gamma R_M T_2}$ the initial speed of sound in the tank.

As previously stated, the flow will transition to purely subsonic when P_t drops below P_{trans}. An approximate but fairly accurate way to deal with the subsonic regime is to continue to use the above solution. However, since eq. (39) overestimates the mass flow rate out of the tank during subsonic flow, it is necessary to compensate by assuming that all flow stops when the tank pressure reaches some stopping pressure, βP_{atm}, rather than P_{atm}. By calculating the exact subsonic-flow solution (which I do not give here), I have empirically determined a simple expression for β:

$$\beta=1.03+0.021\gamma \quad . \tag{41}$$

The time from the beginning to the end of the gas-impulse phase can be estimated by inverting eq. (39) and combining with eq. (41) to give

$$t_{gas} = \tau \left[\left(\frac{P_2}{\beta P_{atm}} \right)^{\frac{\gamma-1}{2\gamma}} - 1 \right] \qquad (42)$$

The end result is that the gas-impulse phase takes several hundredths of a second for an 'open-mouth' 2-liter soda bottle—about the same time as the water-impulse phase.

C-Nozzle Thrust Solution

Again using a combination of the above relations, it is possible to solve eq. (8) for the thrust force generated by the *C* nozzle. This force will be a combination of the pressure force due to the exhaust of high-pressure gas, and of the pure momentum carried by that gas. In the expression given here the force is solely a function of tank absolute pressure:

$$F_{thr} = 2 P_t A_* \left(\frac{2}{\gamma+1} \right)^{\frac{1}{\gamma-1}} - P_{atm} A_* \qquad (43)$$

By integrating the thrust force over time, using eq. (39) for the tank pressure and eq. (42) as the stopping point, we get the total impulse (change in momentum of rocket) due to exhausting gases from a *C* nozzle:

$$I_C = \frac{P_2 V_b}{c_2} \sqrt{\frac{8}{\gamma+1}} \left\{ 1 - \left(\frac{\beta P_{atm}}{P_2} \right)^{\frac{\gamma+1}{2\gamma}} + \frac{P_{trans}}{P_2(\gamma-1)} \left[1 - \left(\frac{P_2}{\beta P_{atm}} \right)^{\frac{\gamma-1}{2\gamma}} \right] \right\} \qquad (44)$$

It is interesting to note that the total impulse does not depend on the size of the opening, A_*. At moderate to high pressures the impulse in eq. (44) is approximately proportional to (P_2 - P_{trans}). In addition, if we neglect air drag the apogee height of a gas-only rocket is proportional to impulse squared. This means that to double the apogee height requires that you at least quadruple the pressure, for a gas-only rocket.

C-D vs. *C* Nozzles

For a given tank pressure there is an optimal area ratio, A_{out}/A_*, for the *CD* nozzle which will maximize thrust. Unfortunately, for water rockets the tank pressure is not constant. A *CD* nozzle will not perform optimally away from its 'design condition', and can even perform substantially worse than a *C* nozzle. Thus, in optimizing the area ratio for a *CD* nozzle it is important to look at the total impulse generated over the course of tank blowdown. Fig. 5 gives the performance of *CD* nozzles, relative to corresponding *C* nozzles, at various area ratios and initial air pressures. Notice that the thrust improvement possible by using a *CD* nozzle is not substantial until very high tank pressures are used.

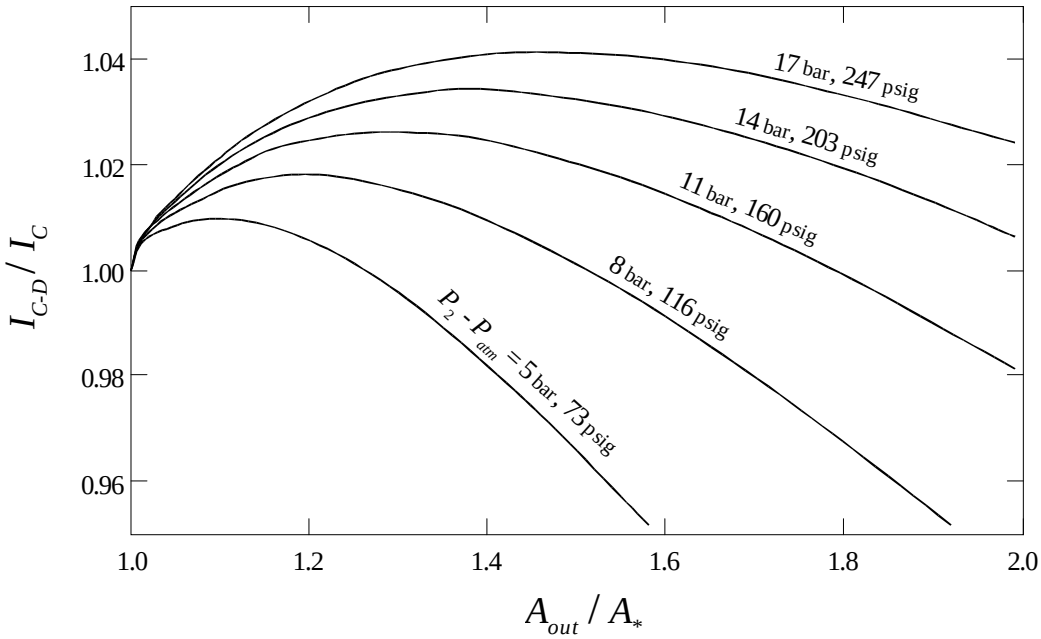

Fig. 5 Impulse efficiency of air tank blowdown through C-D nozzles.

Solution for Rocket Motion

Because of the assumption that the nozzle is quasi-steady state, the F_{int} term can be neglected during gas-impulse. This leaves us with the following equation for rocket acceleration:

$$a = \frac{F_{thr} + F_{drag}}{m_{tot}} - g \qquad (45)$$

The mass of gas will be negligible for typical rocket pressures, nevertheless it is not much trouble to include it in the expression for total mass: $m_{tot} = m_b + \rho_t V_b$. Note that ρ_t can be calculated in time by a combination of eqs. (36) and (39).

Because of the simplifying assumptions we have made for the gas flow, the rocket motion problem is reduced to integrating eqs. (10) with the addition of the auxiliary eqs. (39)-(43) and eq. (45) above. Or, for an even simpler solution, one can treat the gas-impulse as an instantaneous event which increases the rocket's velocity by an amount I_C/m_b, with I_C given by eq. (44).

Appendix A. Effect of Friction

Thus far in this document the effect of friction on the *internal* fluid flow was neglected. Here I give my concept for how to include the effect of friction in the calculations.

Boundary layer theory

The source of energy-robbing friction inside the rocket is due to the fluid having viscosity and interacting with the walls. Because of a condition known as 'no slip' the fluid exactly at the wall of the bottle must have a velocity *u* of zero. Away from the wall, however, the fluid velocity increases in magnitude until it attains a maximum value around the central axis of the bottle. The volume of the bottle is thus divided into two regions: the *boundary layer* region of slow moving fluid next to the wall and the *core* region with full stream velocity. Think of it like the lanes of traffic on a freeway: the boundary layer is the slow lane, and the core is the fast lane.

In the rocket sitting on the launch pad there is no boundary layer: all the fluid has one velocity, namely zero. Following launch the fluid begins accelerating out of the bottle. At first the no-slip condition at the walls slows down only the fluid extremely close to the walls. As time advances, the boundary layer grows or diffuses from the walls into the core region. If the flow out of the bottle continued indefinitely at a constant rate (like the nozzle on a garden hose), the boundary layer would eventually stabilize. It is normally in this regime—steady or fully-developed flow—that the friction factors often given in the fluid mechanics textbooks apply.

For the water-impulse phase the distance the boundary layer extends from the nozzle wall into the flow will vary with time and will be approximately equal to $\sqrt{0.2\,\nu\,t}$, where ν is the kinematic viscosity (shear viscosity divided by density). For water at room temperature $\nu = 9.5\times10^{-7}\ \mathrm{m^2/s}$. This means that in the typical water-impulse time of 0.05 seconds the boundary layer will have only reached 0.1 mm into the flow. *For standard soda-bottles the losses due to a boundary layer will be negligible*. On the other hand, for very narrow or long nozzles the boundary layer can noticeably impede the flow through the nozzle.

For gas flow in the nozzle region the boundary layer can more quickly reach a quasi-steady state in which boundary-layer thickness will mostly depend on distance *z* along the nozzle. In general the boundary layer will have a more significant effect on the gas flow because ν for gases is at least 10 times as large as for water.

Modeling the boundary layer

The presence of the boundary layer has two effects: (1) it obstructs flow because the boundary-layer fluid moves slower than the core fluid, and (2) it produces friction between the boundary layer and the core fluid for the same reason. In both cases the effect can be

approximated by assuming all the fluid has a uniform core velocity but must flow through a reduced effective cross-sectional area.

A relatively simple way to make the approximation is to use the von Karman-Pohlhausen method. I will not explain the method--it is described in nearly all basic fluid mechanics textbooks--except to give my results by applying it to the problem of transient flow through a variable diameter tube.

[section not yet finished]

Appendix B. Effect of Water Condensation

In almost every case there will be water vapor contained in the gas in the rocket. Rocketeers frequently observe the formation of fog in the rocket, which is due to the condensation of that water vapor as the gas temperature drops due to adiabatic expansion. Some have wondered if the energy released in this condensation gives a sizeable kick to rocket thrust.

How much water vapor?

The biggest unknown in analyzing the situation is the mole fraction or volume fraction of water vapor in the gas at liftoff, given by the variable x_0. The initial partial pressure of water vapor in the gas is $x_0 P_0$. Recall that P_0 is the absolute pressure inside the rocket just prior to liftoff. If the vapor water were in equilibrium with any liquid water present in the rocket then

$$x_0 P_0 = P_{vap}(T_0) \tag{B-1}$$

Ordinarily the initial temperature in the rocket (T_0) will be the same as the ambient or atmospheric temperature (T_{atm}). P_{vap} is the equilibrium vapor pressure of water. A reasonably accurate correlation in this temperature regime is:

$$P_{vap}(T) = \exp\left(25.87 - \frac{5310}{T}\right) \tag{B-2}$$

where P_{vap} is in units Pa and T is in K.

I believe $x_0 P_0$ is approximately equal to P_{vap} as given in eq. (B-1), under most conditions.

It may be possible for $x_0 P_0$ in the rocket to exceed P_{vap}, at least temporarily, if a hand pump is used to compress the atmosphere, resulting in supersaturated water vapor in the rocket. If the atmosphere contains some value of relative humidity, $0 < r_{hum} < 1$, the gas in the rocket could have a mole fraction of water vapor as high as:

$$x_0 = r_{hum} \frac{P_{vap}(T_{atm})}{P_{atm}} \tag{B-3}$$

Snow vs. fog

Immediately following liftoff the pressurized gas in the bottle begins expanding and cooling adiabatically. The adiabatic relation between bottle pressure and gas temperature is

$$\frac{P}{P_0}=\left(\frac{T}{T_0}\right)^{\frac{\gamma}{\gamma-1}} \qquad \text{(B-4)}$$

In fact, if the initial temperature T_0 is 298 K (25 °C) the equation above predicts that the gas temperature will drop well below the freezing point of water for essentially all pressurized rockets. For this reason, I believe it is more correct to think of microscopic ice crystals forming in the rocket rather than liquid droplets. Following the boost phase, or after all the pressure energy in the rocket has been expended, the gas in the rocket will warm back up to ambient temperature and these ice crystals will revert first to liquid droplets (normal fog) and then eventually disappear to vapor. The amount of superheating here is relatively minor and so the fog persists for a longer period of time.

When does the vapor condense?

If equilibrium were continually enforced following liftoff, then vapor would condense little by little as the gas cooled. However, there is an energy barrier to the formation of the new phase and so equilibrium can be significantly delayed. The vapor must be 'undercooled' or 'supercooled' to lower the energy barrier so stable nuclei of the new phase can rapidly form. This principle has been known for many years by operators of supersonic wind tunnels: in their case the residence time of the gas is so short that it takes a large degree of supercooling for water to condense in the wind tunnel. I believe that water rockets fit this model because the duration of thrust is also very short.

One observation by operators of wind tunnels is that the water vapor condenses only when there is at least 50°C of supercooling. In my model I make an assumption similar to this: the vapor condenses at precisely 12% supercooling, that is when the absolute temperature is 12% below the equilibrium temperature. This figure is based on my modeling efforts combined with observation of fog formation in rockets using high-speed video.This condition is expressed as

$$x_0 P_{cond} = P_{sub}(T_{cond}/0.88) \qquad \text{(B-5)}$$

where P_{cond} and T_{cond} are the values at the point of condensation and are also related adiabatically by eq. (B-4). Since we are now talking about ice formation, we must use the vapor pressure of ice sublimation as our equilibrium relation. A reasonably accurate function for this is

$$P_{sub}(T)=\exp\left(28.99-\frac{6165}{T}\right) \qquad \text{(B-6)}$$

where, like eq. (B-2), P_{sub} is in units Pa and T in K. A quick solution to eqs. (B-4) and (B-5) is obtained by iterating on the value of T using

$$T = \frac{(0.88)(6165)}{28.99 - \ln(x_0 P_0) - \frac{\gamma}{\gamma - 1}\ln\left(\frac{T}{T_0}\right)} \qquad \text{(B-7)}$$

where again P_0 must be given in units Pa. We let T_{cond} be the solution to T from eq. (B-7) and P_{cond} be the pressure from eq. (B-4) at this point.

Given sufficient undercooling, the energy barrier will be broken. At this point the vapor condenses catastrophically, that is all the water vapor condenses at once. Using this heuristic means that the fog almost instantly forms in the bottle when the gas temperature meets the condition of eq. (B-7). Rapid condensation such as this can be observed in the high-speed videos I have made of rocket launches.

The release of heat

The amount of heat liberated by the condensation (per mole of gas mixture) is

$$Q = H_{sub} x_0 = C_V \Delta T \qquad \text{(B-8)}$$

where H_{sub} is the molar heat of sublimation for H_2O (51.26 kJ/mol) and C_V is the constant-volume heat capacity of the gas mixture. Since the mixture is almost completely made up of the inert component, we can let $C_V = R/(\gamma - 1)$ where R is the ideal gas constant. Not coincidentally, the value 6165 in eq. (B-6) is equal to H_{sub}/R, with implied units of K.

With the above substitutions and a rearrangement of eq. (B-8) we get an expression for the temperature of the gas the moment following condensation

$$T_{new} = T_{cond} + 6165\,(\gamma - 1)\,x_0 \qquad \text{(B-9)}$$

Furthermore, the newly increased pressure is given by

$$P_{new} = P_{cond}\left(\frac{T_{new}}{T_{cond}}\right)(1 - x_0)^{\gamma} \qquad \text{(B-10)}$$

The density of the gas remains constant since there has been no change in mass or volume. The gas is now a heterogeneous mixture, however, and contains a small amount of entrained crystals or droplets.

Results

Using my assumptions and model, let us run some example calculations. Let a rocket be pressurized at room-temperature to 8 bar absolute. Assume the water vapor inside the rocket

is in equilibrium with any liquid present. The partial pressure of water in the pressurized gas mix will then be x_0P_0 = 0.0317 bar. The condensation will occur at an estimated gas temperature of -21 °C and absolute gas pressure of 4.45 bar. The temperature and pressure at that moment are boosted to respectively -11 °C and 4.62 bar.

The end result is that the gas pressure and temperature each will be increased by several percent at the time of vapor condensation. Because the condensation typically occurs pretty late during the gas-impulse phase, it doesn't increase overall thrust more than a few percent because most of the gas has already escaped the rocket and there is not much energy left. Still, it is not much trouble to include the condensation effect in a model. It will provide a small but noticeable kick to the rocket's acceleration.

Appendix C. Finite-Difference Integration

A simple way to get a numerical solution for a set of first-order ordinary differential equations is to use an 'explicit' or 'backward-difference' scheme. For our example we will solve a set of two coupled equations:

$$\frac{dw}{dt} = f(w,x)$$
$$\frac{dx}{dt} = g(w,x)$$

Notice how the changes in w and x (*i.e.* the time derivatives) each depend on both variables. This is what is meant by *coupled* equations.
Now perform the following steps:
(1) Discritize time into small segments, each of size Δt (called the timestep).
(2) Represent each time segment with an integer index k multiplied by the timestep:

$$t_k = k\Delta t$$

The values of w and x at time t_k are given by w_k and x_k, respectively.
(3) Let the initial known values of w and x be w_0 and x_0, respectively.
(4) Now march through time, updating w and x at each successive time index (k = 1, 2, 3, ...) by

$$w_k = w_{k-1} + \Delta t \cdot f(w_{k-1}, x_{k-1})$$
$$x_k = x_{k-1} + \Delta t \cdot g(w_{k-1}, x_{k-1})$$

Notice how w and x are given new values at each time based on their values in the immediately previous time step. This is why this scheme is called backward differencing.

This scheme is easy to implement. For instance, one can create a spreadsheet where each row corresponds to a time segment and has cells for the w and x values which depend (through the above equations) on the corresponding cells for the previous time segment (the row above the given row). Unfortunately, backward differencing can have stability problems (*i.e.* produce inaccurate or erratic results) if Δt is too large and/or if the equations are 'stiff.' There are better, more stable schemes such as 'predictor-corrector' and Runge-Kutta, however they are a little more difficult to program.

Appendix D. Computational Examples

[section not yet finished]