Part I

Linear sloshing dynamics

1

Fluid field equations and modal analysis in rigid containers

1.1 Introduction

The theory of liquid sloshing dynamics in partially filled containers is based on developing the fluid field equations, estimating the fluid free-surface motion, and the resulting hydrodynamic forces and moments. Explicit solutions are possible only for a few special cases such as upright cylindrical and rectangular containers. The boundary value problem is usually solved for modal analysis and for the dynamic response characteristics to external excitations. The modal analysis of a liquid free-surface motion in a partially filled container estimates the natural frequencies and the corresponding mode shapes. The knowledge of the natural frequencies is essential in the design process of liquid tanks and in implementing active control systems in space vehicles. The natural frequencies of the free liquid surface appear in the combined boundary condition (kinematic and dynamic) rather than in the fluid continuity (Laplace's) equation.

For an open surface, which does not completely enclose the field, the boundary conditions usually specify the value of the field at every point on the boundary surface or the normal gradient to the container surface, or both. The boundary conditions may be classified into three classes (Morse and Fesbach, 1953):

(1) the *Dirichlet boundary conditions*, which fix the value of the field on the surface;
(2) the *Neumann boundary conditions*, which fix the value of the normal gradient on the surface; and
(3) the *Cauchy conditions*, which fix both value of the field and normal gradient on the surface.

Each class is appropriate for different types of equations and different boundary surfaces. For example, Dirichlet conditions on a closed surface uniquely specify a solution of Laplace's equation inside the closed surface.

The variational formulation based on Hamilton's principle is regarded as the most powerful tool for developing the fluid field equations. This approach has been proposed and used by Lawrence, *et al*. (1958), Troesch (1960), Bogoryad (1962), Borisova (1962), Petrov (1962a,b,c), Moiseev (1964), Moiseev and Petrov (1966), Luke (1967), Whitham (1967), Lukovskii (1967, 1976), Moiseev and Rumyantsev (1968), Limarchenko (1978a, 1980, 1983b), Lukovskii and Timokha (1992, 1995), and Rocca, *et al*. (1997). The method of integral equations was adopted for containers whose wetted walls are not straight vertical but curved such as spherical containers and horizontal cylindrical containers (see, e.g., Budiansky, 1960 and McIver, 1989). Some analytical and approximate approaches to estimating the sloshing frequencies were developed by Housner (1963a), Evans (1990), and Evans and Linton (1993).

Henrici, *et al*. (1970) presented an extensive treatment of liquid sloshing in a half-space bounded above by a rigid plane that contains either a circular or infinite-strip aperture. They obtained both upper and lower bounds of the natural frequencies. Troesch and Troesch (1972) and Miles (1972) discussed some features of the spectrum of the eigenvalues of liquid sloshing in a half-space with an emphasis on their upper bounds. Banning, *et al.* (1966) built an apparatus for demonstrating the dynamics of liquid sloshing.

The dynamic behavior of liquid propellant free surface was addressed by Ehrlich (1959), Abramson (1961b, 1965), Eulitz and Glaser (1961), Eulitz (1963), Bonneau (1964), Buchanan and Bugg (1966), Fontenot (1968), Martin (1971), and Dodge and Garza (1971). The modal analysis in a circular cylindrical container was originally treated by Poisson (1828) but the results were not interpreted because the theory of Bessel's function was not sufficiently developed at that time. The equations of motion of a liquid in rigid rectangular and rigid circular tanks of uniform depth and with linearized boundary conditions were also given by Rayleigh (1887), Steklov (1902), and Lamb (1945). The solution of the Laplace equation using the method of separation of variables is somewhat less powerful for cases where the liquid depth is variable and other methods, such as the Ritz method, should be used. Bratu (1971) studied the oscillations of liquid masses in reservoirs.

The free-surface mode shapes for containers with axial symmetry were determined by Borisova (1962), Bonneau (1964), Moiseev and Petrov (1965, 1966), Pfeiffer (1967a,b), Einfeldt, *et al.* (1969), McNeil and Lamb (1970), Henrici, *et al.* (1970), Pshenichnov (1972), and Boyarshina and Koval'chuk (1986) determined the normal modes and natural frequencies of the free surface in an inclined cylinder. Trotsenko (1967) studied the liquid oscillations in a cylindrical tank with annular baffle. For a spherical tank, the problem is analytically more complex and approximate solutions for the natural frequencies were obtained by Bauer (1958a), Budiansky (1960), Leonard and Walton (1961), Riley and Trembath (1961), Lukovskii (1961a,b), Chu (1964a), Boudet (1968), McIver (1989), El-Rahib and Wagner (1981), and Bauer and Eidel (1989b). The natural frequency of horizontal circular canals and spherical containers is determined from an integral equation, which is usually discretized into a matrix form for numerical calculations (Barnyak, 1997).

Bauer (1964b) and Mooney, *et al.* (1964a,b) analyzed the free-surface oscillations in a quarter tank and in a tank with annular sector cross-section. In both cases, the natural frequency of the free surface was found to have the same expression as the circular cylindrical tank but with different roots of the Bessel function. The liquid sloshing frequencies for different container geometries were evaluated by Miles (1964, 1972), Kuttler and Sigillito (1969), Fox and Kuttler (1981, 1983), Meserole and Fortini (1987), and McIver and McIver (1993). The influence of movable devices and internal pipes on the natural frequencies of the free surface was determined by Siekmann and Chang (1971b) and Drake (1999). Bauer and Eidel (1999c) considered different configurations of cylindrical containers.

Based on the two-dimensional analysis of liquid motion in rectangular tanks, the natural frequency depends essentially on the liquid depth to width ratio. The effect of liquid depth is diminished as the mode order increases. Graham and Rodriguez (1952) solved the three-dimensional velocity potential for which the natural frequency depends on the three major dimensions of the fluid. Ghali (1965) determined the nonlinear dependence of the natural frequencies on the wave motion amplitude. The influence of damping on the natural frequency was studied experimentally by Ghali (1965), Scarsi and Brizzolara (1970), Scarsi (1971), and Schilling and Siekmann (1980). It was found that for higher viscosities (of kinematic viscosity $\nu = 2.5$ poise) the resonance frequency is slightly higher than the predicted value for an ideal liquid.

The purpose of this chapter is to develop the fluid field equations with reference to an inertial frame and with respect to a moving coordinate system. The variational approach is demonstrated as a tool to derive the boundary-value problem in one treatment. The modal analysis of liquid free-surface motion is formulated for different tank shapes. The analysis includes the estimation of the velocity potential function, fluid free-surface natural frequencies and mode shapes. The influence of surface tension is included in some cases. However, its effect is dominant in a microgravity field as will be demonstrated in Chapter 12. Note that analytical solutions in a closed form are only obtained for regular tank shapes whose walls are upright straight. For other tank geometries with variable depth, one can determine the natural frequencies and mode shapes either experimentally or numerically.

1.2 Fluid field equations

The analytical description of the fluid field equations is documented for different cases of tank geometries by Ewart (1956), Bauer (1962a, 1966c, 1969a), Lomen (1965a), Abramson (1966a), Ibrahim (1969), Khandelwal (1980), Kornecki (1983), and Bauer (1999). The general equations of motion for a fluid in closed containers can be simplified by assuming the container rigid and impermeable. Furthermore, the fluid is assumed inviscid, incompressible, and initially irrotational. Capillary or surface tension effects will be ignored in a gravitational field. However, the effect of surface tension will be introduced for some simple cases. The free-surface oscillations can be generated by giving an initial impulse, or an initial disturbance to the free surface.

This section considers the general case of a tank moving along some trajectory in space. The formulation is applicable to free and forcing liquid free-surface oscillations. It is convenient to refer the fluid motion to a moving coordinate system as the variables are measured relative to the moving frame. It is also useful to write the fluid equations of motion with reference to stationary and moving coordinates as shown in Figure 1.1. In the present analysis, the tank is allowed to move in planar curvilinear motion without rotation. Let $O'X'Y'Z'$ be the stationary Cartesian coordinate frame. The Euler equations of motion of the fluid are written in the vector form

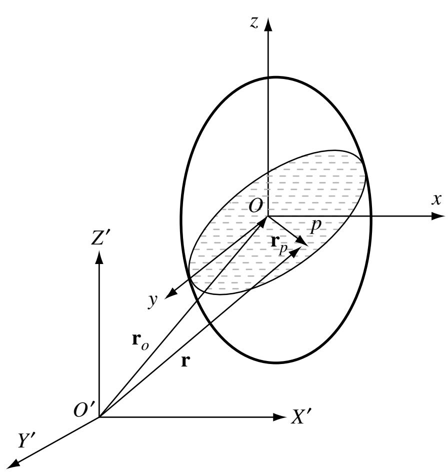

Figure 1.1 Moving liquid container showing inertia and moving coordinates.

$$\frac{\partial}{\partial t}\boldsymbol{q} + (\boldsymbol{q}\cdot\nabla)\boldsymbol{q} = -\frac{1}{\rho}\nabla P - \nabla(gZ') \tag{1.1}$$

where $\boldsymbol{q}$ is the fluid velocity, $\partial\boldsymbol{q}/\partial t$ is the local acceleration of the flow at the point whose coordinates are not allowed to vary (this acceleration is measured by a fixed observer), $(\boldsymbol{q}\cdot\nabla)\boldsymbol{q}$ is the convective acceleration for a fluid particle drifting with the stream at a velocity $\boldsymbol{q}$ in the flow direction (this acceleration is measured by an observer moving with the particle p), P is the fluid pressure, ρ is the fluid density, $gZ^{'}$ is the gravitational potential, and ∇ is an operator given for different coordinate frames in the appendix to this chapter. Note that the convective acceleration $(\boldsymbol{q}\cdot\nabla)\boldsymbol{q}$ may also be written in the form (Thomson, 1965, p. 44)

$$(\boldsymbol{q}\cdot\nabla)\boldsymbol{q} = \frac{1}{2}\nabla q^2 - \boldsymbol{q}\times(\nabla\times\boldsymbol{q}) = \frac{1}{2}\nabla q^2 \tag{1.2}$$

For irrotational flow the curl of the velocity vanishes, that is, $\nabla\times\boldsymbol{q} = 0$.

For irrotational fluid motion, there exists a velocity potential function, Φ, whose (negative) gradient gives the fluid velocity,

$$\mathbf{q} = -\nabla\Phi \tag{1.3}$$

The negative sign is optional and in some cases it may be removed provided the analysis preserves the sign convention. Introducing relations (1.2) and (1.3) into equation (1.1) gives

$$\nabla\left(\frac{P}{\rho} + \frac{1}{2}q^2 + gZ' - \frac{\partial\Phi}{\partial t}\right) = 0 \tag{1.4}$$

Upon integrating equation (1.4) one obtains

$$\frac{P}{\rho} + \frac{1}{2}q^2 + gZ' - \frac{\partial\Phi}{\partial t} = C(t) \tag{1.5}$$

where $C(t)$ is an arbitrary function of time.

Equation (1.5) is the general form of Kelvin's equation for an unsteady fluid flow. In this equation the potential function Φ is a function of space and time, and its derivative with respect to time measures the unsteadiness of the flow. However, $\partial\Phi/\partial t$ is interpreted as the work done on a unit mass of the fluid whose coordinates are (X, Y, Z). Furthermore, equation (1.5) is only valid for incompressible flow for which the continuity condition $\nabla\cdot\boldsymbol{q} = 0$ yields Laplace's equation, which after introducing equation (1.2) takes the form

$$\nabla^2\Phi = 0 \tag{1.6}$$

Let $Oxyz$ be another coordinate frame fixed to the tank such that the Oxy plane coincides with the undisturbed free surface. Let $\boldsymbol{V}_0$ be the velocity of the origin O relative to the fixed origin O'. In this case, the time rate of change of the velocity potential Φ at a point fixed in the stationary frame $O'X'Y'Z'$ as measured by an observer in the moving frame $Oxyz$ is $(\partial/\partial t - \boldsymbol{V}_0\cdot\nabla)\Phi$, since this point will appear to have a velocity $-\boldsymbol{V}_0$ with respect to the observer. Accordingly, the pressure equation (1.5) takes the form

$$\frac{P}{\rho} + \frac{1}{2}q^2 + gZ' - \frac{\partial\Phi}{\partial t} + \boldsymbol{V}_0\cdot\nabla\Phi = C(t) \tag{1.7a}$$

The fluid particle velocity $\boldsymbol{q}_{\text{rel}}$ relative to the moving coordinate is

$$\boldsymbol{q}_{\text{rel}} = \boldsymbol{q} - \boldsymbol{V}_0 = -\nabla\Phi - \boldsymbol{V}_0 \tag{1.8}$$

Expressing $\boldsymbol{q}$ in terms of $\boldsymbol{q}_{\text{rel}}$ and $\boldsymbol{V}_0$, using relation (1.8), gives

$$\frac{P}{\rho} + \frac{1}{2}q_{\text{rel}}^2 + gZ' - \frac{\partial\Phi}{\partial t} - \frac{1}{2}V_0^2 = C(t) \tag{1.7b}$$

Equation (1.7a) is written in terms of the total fluid velocity as measured by the fixed coordinate and equation (1.7b) is given in terms of the fluid relative velocity to the tank moving coordinates. At the free surface, the pressure is equivalent to the ambient pressure or can be set to zero in equation (1.7a). This gives the dynamic boundary condition

$$\frac{1}{2}(\nabla\Phi \cdot \nabla\Phi) + g\eta - \frac{\partial\Phi}{\partial t} + \mathbf{V}_0 \cdot \nabla\Phi = 0 \tag{1.9}$$

where the function $C(t)$ has been absorbed in the potential function, Φ.

The vertical velocity of a fluid particle located on the free surface $z = \eta(r, \theta, t) = \eta(x, y, t)$ should be equated to the vertical velocity of the free surface itself. This condition is known as the kinematic free-surface condition and is given by the following expression

$$-\frac{\partial\Phi}{\partial z} = \frac{\partial\eta}{\partial t} + \mathbf{q}_{\text{rel}} \cdot \nabla\eta \tag{1.10}$$

At the wetted rigid wall and bottom, the velocity component normal to the boundary must have the same value of the corresponding velocity component of the solid boundary at the point in question. For example, if the tank is allowed to move in the vertical plane then the velocity vector in terms of Cartesian and cylindrical coordinates may be written in the form, respectively

$$\mathbf{V}_0 = \dot{X}_0\boldsymbol{i} + \dot{Z}_0\boldsymbol{k} \tag{1.11a}$$

$$\mathbf{V}_0 = \left(\dot{X}_0\cos\theta\right)\mathbf{i}_r - \left(\dot{X}_0\sin\theta\right)\mathbf{i}_\theta + \dot{Z}_0\mathbf{i}_z \tag{1.11b}$$

The boundary conditions at the wall and bottom for Cartesian and cylindrical coordinates are, respectively,

$$-\left.\frac{\partial\Phi}{\partial z}\right|_{z=-h} = \dot{Z}_0, \quad -\left.\frac{\partial\Phi}{\partial x}\right|_{x=a} = \dot{X}_0 \tag{1.12a}$$

$$-\left.\frac{\partial\Phi}{\partial z}\right|_{z=-h} = \dot{Z}_0, \quad -\left.\frac{\partial\Phi}{\partial r}\right|_{r=R} = \dot{X}_0\cos\theta \tag{1.12b}$$

It is possible to split the total velocity potential function, Φ, into a disturbance potential function, $\tilde{\Phi}$, and a potential function, Φ_o, which defines the motion of the tank, that is,

$$\Phi = \tilde{\Phi} + \Phi_o \tag{1.13}$$

The function Φ_o can be determined by integrating equation (1.11) as

$$\Phi_o = -\dot{X}_0 r\cos\theta - \dot{Z}_0 z - \frac{1}{2}\int (\dot{X}_0^2 + \dot{Z}_0^2)\, dt \tag{1.14}$$

Introducing (1.13) and (1.14) into the free-surface boundary conditions gives

$$\frac{1}{2}(\nabla\tilde{\Phi}\cdot\nabla\tilde{\Phi}) + (g+\ddot{Z}_0)\eta - \frac{\partial\tilde{\Phi}}{\partial t} + \ddot{X}_0 r\cos\theta = 0 \tag{1.15}$$

$$-\frac{\partial\tilde{\Phi}}{\partial z} = \frac{\partial\eta}{\partial t} - \frac{\partial\eta}{\partial r}\frac{\partial\tilde{\Phi}}{\partial r} - \frac{1}{r^2}\frac{\partial\eta}{\partial\theta}\frac{\partial\tilde{\Phi}}{\partial\theta} \tag{1.16}$$

The corresponding conditions for a rectangular tank are

$$\frac{1}{2}(\nabla\tilde{\Phi}\cdot\nabla\tilde{\Phi}) + (g+\ddot{Z}_0)\eta - \frac{\partial\tilde{\Phi}}{\partial t} + \ddot{X}_0 x = 0 \tag{1.17}$$

$$-\frac{\partial\tilde{\Phi}}{\partial z} = \frac{\partial\eta}{\partial t} - \frac{\partial\eta}{\partial x}\frac{\partial\tilde{\Phi}}{\partial x} - \frac{\partial\eta}{\partial y}\frac{\partial\tilde{\Phi}}{\partial y} \tag{1.18}$$

One may introduce the effect of surface tension, σ, by including the pressure change across the displaced free liquid surface as described by the Laplace–Young equation

$$p_s = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{1.19}$$

where R_1 and R_2 are the principal radii of curvature. The complete formulation of the boundary value problem in terms of the disturbance potential function is summarized as follows:

(1) *for a cylindrical container*:

$$\nabla^2\tilde{\Phi} = 0 \tag{1.20a}$$

$$\left.\frac{\partial\tilde{\Phi}}{\partial r}\right|_{r=R} = 0,\quad \left.\frac{\partial\tilde{\Phi}}{\partial z}\right|_{z=-h} = 0 \tag{1.20b, c}$$

$$\frac{1}{2}(\nabla\tilde{\Phi}\cdot\nabla\tilde{\Phi}) + (g+\ddot{Z}_0)\eta - \frac{\partial\tilde{\Phi}}{\partial t} + \frac{\sigma}{\rho}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) + \ddot{X}_0 r\cos\theta = 0, \quad \text{at } z = \eta(r,\theta,t) \tag{1.20d}$$

$$-\frac{\partial\tilde{\Phi}}{\partial z} = \frac{\partial\eta}{\partial t} - \frac{\partial\eta}{\partial r}\frac{\partial\tilde{\Phi}}{\partial r} - \frac{1}{r^2}\frac{\partial\eta}{\partial\theta}\frac{\partial\tilde{\Phi}}{\partial\theta}, \quad \text{at } z = \eta(r,\theta,t) \tag{1.20e}$$

The curvature κ for cylindrical coordinates is given by the expression

$$\begin{aligned}\kappa &= -\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\\ &= -\frac{\eta_{rr}\left(1+(\eta_\theta^2/r^2)\right) + (1+\eta_r^2)\left((\eta_r/r)+(\eta_{\theta\theta}/r^2)\right) - 2\,\eta_r\,(\eta_\theta/r^2)(\eta_{rr}+(\eta_\theta/r))}{\left[1+\eta_r^2+(\eta_\theta^2/r^2)\right]^{3/2}}\end{aligned} \tag{1.21}$$

This expression can be linearized in the form

$$\kappa = -\left[\eta_{rr} + \frac{\eta_r}{r} + \frac{\eta_{\theta\theta}}{r^2}\right] \tag{1.22}$$

(2) *for a rectangular container*:

$$\nabla^2 \tilde{\Phi} = 0 \tag{1.23a}$$

$$\left.\frac{\partial \tilde{\Phi}}{\partial x}\right|_{x=\pm a/2} = 0, \quad \left.\frac{\partial \tilde{\Phi}}{\partial y}\right|_{x=\pm b/2} = 0, \quad \left.\frac{\partial \tilde{\Phi}}{\partial z}\right|_{z=-h} = 0 \tag{1.23b, c, d}$$

$$\frac{1}{2}\left(\nabla\tilde{\Phi}\cdot\nabla\tilde{\Phi}\right) + (g + \ddot{Z}_0)\eta - \frac{\partial \tilde{\Phi}}{\partial t} + \frac{\sigma}{\rho}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + \ddot{X}_0 x = 0, \quad \text{at } z = \eta(x, y, t) \tag{1.23e}$$

$$-\frac{\partial \tilde{\Phi}}{\partial z} = \frac{\partial \eta}{\partial t} - \frac{\partial \eta}{\partial x}\frac{\partial \tilde{\Phi}}{\partial x} - \frac{\partial \eta}{\partial y}\frac{\partial \tilde{\Phi}}{\partial y}, \quad \text{at } z = \eta(x, y, t) \tag{1.23f}$$

The curvature, κ, for cylindrical coordinates is given by the expression

$$\kappa = -\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = -\frac{\eta_{xx}\left(1 + \eta_y^2\right) + \eta_{yy}(1 + \eta_x^2) - 2\eta_x\eta_y\eta_{xy}}{\left[1 + \eta_x^2 + \eta_y^2\right]^{3/2}}. \tag{1.24}$$

This expression can be linearized in the form

$$\kappa = -\left[\eta_{xx} + \eta_{yy}\right] \tag{1.25}$$

For other container geometries, such as spherical, prolate and oblate spherical, and elliptic containers, the continuity equation and other related operators are listed in the appendix to this chapters. Note that the velocity potential function, $\tilde{\Phi}$, must satisfy Laplace's equation, $\nabla^2\tilde{\Phi} = 0$, which is a linear partial differential equation. The nonlinearity in the boundary value problem only exists in the free-surface boundary conditions on $z = \eta$. If one is interested in the modal analysis then one should drop the nonlinear and nonconservative terms from the free-surface boundary conditions. If the potential function is obtained analytically in a closed form, then the natural frequencies of the fluid free surface are obtained by using the dynamic free-surface condition based on the fact that $\tilde{\Phi}$ is harmonic in time. Another powerful approach is to use the variational formulation together with the Rayleigh–Ritz method. The next section describes an alternative approach based on the variational principle.

1.3 Variational formulation

The variational approach is based on establishing the superlative of a certain function that describes the system behavior. The Lagrangian, $L = T - V$, has to be minimized (or maximized), where T and V are the kinetic and potential energies of the system, respectively. The variational principle, or Hamilton's principle, is

$$\delta I = \delta \int_{t_1}^{t_2} (T - V)\, dt = 0 \tag{1.26}$$

Basically Hamilton's principle states that the actual path in the configuration space yields the value of the definite integral stationary with respect to all arbitrary variations of the path between two instants of time t_1 and t_2 provided the path variations vanish at these two end points. For any actual motion of the system, the system will move so that the time average of the difference between the kinetic and potential energies will be a minimum. This formulation is very powerful since it brings in one statement the fluid field equations and the associated boundary conditions. Substituting for the kinetic energy, $T = \int_{\upsilon} (\rho/2)|\nabla\Phi|^2 \, d\upsilon$, and potential energy, $V = \int_S (\eta/2)(\rho g \eta \, dS)$, where υ is the fluid volume, and S is the fluid free surface, in equation (1.26), and taking the variation gives

$$\delta I = \delta \int_{t_1}^{t_2} dt \left\{ \int_{\upsilon} \frac{\rho}{2} |\nabla\Phi|^2 \, d\upsilon - \int_S \frac{\eta}{2} (\rho g \eta \, dS) \right\}$$

$$= \rho \int_{t_1}^{t_2} dt \left\{ \int_{\upsilon} \nabla\Phi \nabla\delta\Phi \, d\upsilon - g \int_S \eta\delta\eta \, dS \right\} = 0 \tag{1.27}$$

The volume integral can be transformed into a surface integral using Green's formula, $\int_{\upsilon} \nabla\Phi\nabla\delta\Phi \, d\upsilon = \int_S \Phi \frac{\partial\delta\Phi}{\partial n} \, dS$. Furthermore, one may use the relationship

$$\nabla\Phi = \boldsymbol{n}\frac{\partial\Phi}{\partial n} = -\boldsymbol{n}\frac{\partial\eta}{\partial t}$$

(see, e.g., Thomson, 1965), where $\boldsymbol{n}$ is the unit vector along the normal at the point in question to the equi-potential surface of Φ. In this case, the variational takes the form

$$\rho \int_{t_1}^{t_2} dt \left\{ \int_S \left\{ \Phi \frac{\partial\delta\Phi}{\partial n} - g\eta\delta\eta \right\} dS = -\rho \int_{t_1}^{t_2} dt \, \left\{ \int_S \left\{ \Phi\delta\frac{\partial\eta}{\partial t} + g\eta\delta\eta \right\} dS = 0\right.\right.$$

Integrating by parts gives

$$\rho \int_{t_1}^{t_2} \int_S \left\{ -\frac{\partial\Phi}{\partial t} + g\eta \right\} \delta\eta \, dS \, dt = 0$$

This statement yields the linearized dynamic free surface condition

$$-\frac{\partial\Phi}{\partial t} + g\eta = 0 \tag{1.28}$$

Moiseev and Rumyantsev (1968) introduced the Neumann operator H, which makes the velocity potential, Φ, harmonic inside the fluid volume domain, υ. The harmonic property is based on the fact that the integral of free-surface velocity, $\dot{\eta}(s)$, vanishes over the free surface, that is, $\int_S \dot{\eta}(s) \, ds = 0$. The function Φ satisfies the following conditions on the fluid boundary $\frac{\partial\Phi}{\partial n} = 0$ at the container walls, and $\frac{\partial\Phi}{\partial n} = -\dot{\eta}$ at the free surface.

In this case, one can write the correspondence as $\Phi = H\dot{\eta}$, where H is the integral operator

$$H\dot{\eta} = \int_S H(s, \upsilon)\dot{\eta}(s)\; ds = \Phi(\upsilon) \tag{1.29}$$

Note that the kernel is the Green function of Neumann's problem for the region υ. This kernel is known to be symmetric (see, e.g., Mikhlin, 1964, and Gunter, 1965). The kernel is logarithmic in the case of the plane problem, that is, when the volume is reduced to a plane surface. The kernel is a polar function in the case of the three-dimensional problem. Thus, H is a fully continuous selfadjoint operator, and one can write the following representation

$$\Phi(\upsilon) = H\frac{\partial \Phi}{\partial z} = -H\frac{\partial \eta}{\partial t} \tag{1.30}$$

Taking the time derivative for both sides and using equation (1.28), gives

$$g\eta + H\frac{\partial^2 \eta}{\partial t^2} = 0 \tag{1.31}$$

The average energy function can be written in terms of the Neumann operator in the form

$$I_1 = \frac{\rho}{2}\int_{t_1}^{t_2}\int_S \left\{ H\left(\frac{\partial \eta}{\partial t}\right)^2 - g\eta^2 \right\} ds\, dt \tag{1.32}$$

Alternatively, one can write the average energy function in terms of the scalar potential function in the form, after using equation (1.28)

$$I = \int_{t_1}^{t_2} dt \left\{ \int_\upsilon \frac{\rho}{2} |\nabla \Phi|^2 \; d\upsilon - \frac{1}{g}\int_S \Phi^2 \; ds \right\} \tag{1.33}$$

Equations (1.32) and (1.33) can be used for estimating the natural frequencies for the liquid free surface. At the free surface, S, both the velocity potential function and the free-surface wave height can be expressed in terms of time and space as follows

$$\Phi(s, t) = F(s)\cos \omega t, \qquad \text{and} \qquad \eta(s, t) = G(s)\sin \omega t \tag{1.34}$$

where ω is the free-surface natural frequency. Substituting equations (1.34) in the average energy functions given by equations (1.32) and (1.33) and integrating over time t from $t_1 = 0$ to $t_2 = 2\pi/\omega$, gives

$$I_1 = \lambda \int_S HG \cdot G\, ds - \int_S G^2\, ds \tag{1.35}$$

$$I_2 = \int_\upsilon \frac{\rho}{2} |\nabla F|^2\, d\upsilon - \lambda \int_S F^2\, ds \tag{1.36}$$

where $\lambda = \omega^2/g$. The natural frequencies of the fluid free surface are determined by using the Rayleigh–Ritz method. The method is based on introducing a linear combination of a