Derivation of Tsiolkovski’s Rocket Formula.

Imagine a rocket of mass ‘M’ and velocity ‘u’ relative to a stationary observer. Then let ‘v’ be the exhaust velocity of the rocket jet as it leaves the nozzle. Now if a small amount of fuel burns in the combustion chamber of the rocket motor. It produces a high energy gas stream which expands and leaves the the rocket nozzle to enter the atmosphere. We can say that the rocket mass has been effectively reduced by a small amount $dM$ whilst the exhaust jet has been increased by the same amount $dM$.

We can then create an expression for the momentum of the rocket and the exhaust jet after a small increment in time $dt$:

$$\Delta_{\text{Rocket}} = (M - dM)du$$  \hspace{1cm} (1)

$$\Delta_{\text{Exhaust}} = dM(u + v)$$  \hspace{1cm} (2)

Applying Newton’s law for the ‘Conservation of momentum’ to equations 1 and 2 we have.

$$dM(u + v) + du(M - dM) = 0$$  \hspace{1cm} (3)

Initially the gas stream jet velocity will be far greater than that of the rocket. Therefore

$$u \ll v \quad \text{So we say } u \Rightarrow 0 \quad \text{tends to zero or is insignificant relative to the size of velocity } v$$

and if the elemental mass of exhaust gas lost relative to that of the whole rocket very small. Then we can say

$$M \gg dM \quad \text{or } dM \Rightarrow 0.$$  \hspace{1cm} (4)

Re writing equation 3 we get

$$dM.v + du.M = 0$$  \hspace{1cm} (4)

Rearranging Eq4 for $du$

$$du = -\frac{v.dM}{M}$$  \hspace{1cm} (5)

Integrating between definite limits from the Initial point 1 to point 2 after an elemental time $dt$ when $du$ and $dM$ occur. We get

$$\int_{u_1}^{u_2} 1.du = -\frac{v}{M_1} \int_{m_1}^{m_2} dM'$$  \hspace{1cm} (6)

$$[ u_1 - u_2 ] = -v [ \ln M_1 - \ln M_2 ] = -v \cdot \ln \frac{M_1}{M_2}$$

Multiply both sides of the equation by –1 and rearranging for $u_2$

$$u_2 = v \cdot \ln \frac{M_1}{M_2} + u_1$$

*The Water Rocket Explorer*