The Tsiolkovski End function.

Derivation of Tsiolkovski's adapted for low mass high acceleration projectiles.

$$\Delta_{\text{Projectile}} = (M - dM)du \quad 1$$

$$\Delta_{\text{Exhaust}} = dM(u + v) \quad 2$$

Applying Newtons law for the conservation of momentum to Equations 1 and 2.

$$dM(u + v) + (M - dM)du = 0 \quad 3$$

In the normal derivation at the limit we use the convention

$$M.>> dM \qquad dM \rightarrow 0$$

And $\quad v >> u \qquad u \rightarrow 0$

For the case of light projectiles and water rockets the impulse phase can be split into two distinct phases.

Before the function $\text{Ln}\,\dfrac{M1}{M2} = 1$ or $\dfrac{M1}{M2} = e^{1}$

Which I refer to as the Tsiolkovski point t_{Tsiol} and after.

At this limit $u_2 = v + u_1$

So the limit case of $u \rightarrow 0$ is now inappropriate

$$u_2 = v.\text{Ln}\,\frac{M1}{M2} + u_1 \quad 4$$

Beyond this point and for the remainder of the jet impulse Δit
Another set of conditions need to be selected to both match and model the actual end function.
The acceleration beyond the Tsiolkovski point.

$$\Delta it_2 . e^{\frac{mg1}{mg2}} = \Delta a_2 . g \quad 5$$

$$e^{\frac{mg1}{mg2}} = \frac{\Delta a2}{\Delta it2}.g$$

Using the relationship $\qquad x = e^{y}$

Then $\qquad y = \text{Log}_e\, x$

Modified mass ratio $\qquad \dfrac{mg1}{mg2} = \text{Ln}\,\dfrac{\Delta a2}{\Delta it2}.g$

Where

Δit_2 is the impulse time left to complete the mass flow.

Δa_2 is the change in acceleration of the projectile between t_{Tsiol} or (t_1) and the completion of the impulse at t_2.

$mg1$ is the mass at the Tsiolkovski point modified to take account of relativity.

$mg2$ is the mass at the end of the jet impulse phase modified to take account of relativity.

g is the standard reference gravitational acceleration at standard reference time relative to an external observer.

Hypothesis :

If time is related to a gravitation field and this graviton field is created by accelerating a mass. Then extreme acceleration of the mass would deform the time regime of that mass relative to an static observer.
Imagine a small mass of 200gms accelerated to 135g after time 0.05sec
Then this mass is progressively reduced to 50gms and accelerated to 450g over a very short time interval t = 0.01sec.

Worked Example :

Using data measured at 500 images /sec for a basic bluff Badoit bottle with fins.
(No nose-cone)

Conditions at the Tsiolkovski point :

M1 = 0.204 Kgs

a_1 = 135.9g

t_{Tsiol} or (t_1) = 0.0492sec

$m.\frac{d}{dt}$ = 12.Kgms/sec

u_1 = 45 m/s

$\frac{mf}{tTsiol} \geq 220$ 7 and 8bar

Conditions at the end of the jet impulse

M2 0.057Kgs

a_2 = 448.22g

t_2. = 0.0665 sec
Between points 1 and 2.

$\Delta a_2 = a_2 - a_1$

$\Delta a_2 =$ 448.22g - 135.9g

$\Delta a_2 =$ 312.32 g

Δit_2 = 0.0173 sec

The assumption here is that above a threshold specific mass flowrate of $\frac{mf}{tTsiol} \geq 220$

The accelerated mass has experiences a relativity related time reduction. Induced by the gravitational field generated by the accelerated mass. The extreme acceleration seen during the later part of the impulse can then be treated by effectively reducing the relative equivalent mass by the equivalence by which time is slowed due to the acceleration.
Imagine time effectively slowing down relative to a static observer, due to the mass being accelerated . So that both the mass and time have momentarily been effectively reduced, relative to a static observer. This change would be proportional to the acceleration of the mass.

$$mg_1 = \frac{InitialMass}{\Delta aTrelative.}$$

$$mg_1 = \frac{0.2049}{135.9} = 0.001507$$

$mg_1 = 1.507.10^{-3}$

$$mg_2 = \frac{InitialMass}{\Delta aTrelative.}$$

$$mg_2 = \frac{0.057}{448.22} = 0.000124$$

$mg_2 = 1.24.\ 10^{-4}$

$$\frac{mg1}{mg2} = \frac{1.508.10^{-3}}{1.249.10^{-4}} = 1.2068.10^{1}$$

$$e^{\frac{mg1}{mg2}} = e^{12.068}$$

$e^{\frac{mg1}{mg2}} = 174160.639$

Then

$\Delta it_2 .\ e^{\frac{mg1}{mg2}} = 0.0165 * 174160.64 = 3012.98$

$\Delta it_2 .\ e^{\frac{mg1}{mg2}} = 3012.98$

$\Delta a_2 = 312.32\ g = 3063.86$

Error 1.6%

Allowing for experimental error then this model is possible.

Dimensional Analysis

$$\Delta it_2 . e^{\frac{mg1}{mg2}} = \Delta a_2 . g$$

Assumptions :

Δit_2 Term is the impulse of the remaining jet pulse. $[T_{Earth}]$

Δa_2 Is the change in acceleration between t_{Tsiol} or (t_1) and t_2. $\left[\frac{L}{T^2_{Earth}}\right]$

$e^{\frac{mg1}{mg2}}$ Is the rate of change of mass ratio relative to the local time.

If the local time changes relative to t_{Earthg} then this will change the local gravitational constant g_{local}.

Giving a dimensional parameter of the form $\left[\frac{M_{ratio}}{T_{local}}\right]\left[\frac{L}{T^2_{Earth}}\right]$

$$\Delta it_2 . e^{\frac{mg1}{mg2}} = \Delta a_2 . g$$

$$[T_{Earth}]\left[\frac{M_{ratio}}{T_{local}}\right]\left[\frac{L}{T^2_{Earth}}\right] = \left[\frac{L}{T^2_{Earth}}\right]$$

$$M_{ratio} = 1$$

$e^{\frac{mg1}{mg2}}$ Represents the gravitational change of a rapidly accelerated mass.

Note : At accelerations of the order of 150 > 450g_{Earth}

John Gwynn *The Water Rocket Explorer*