Application of Tsiolkovski Formula to multi-stage rocket launcher systems.

Comparison of multi stage with single stage.

Lets take a rocket at launch and analyse the proportion of mass taken by each of its principle parts.
At launch whether the rocket is a single or multiple stage rocket the mass can be represented by

$$M_{Launch} = m_{struct} + m_{fuel} + m_{sat} \qquad 1$$

Now let the fuel used during the first stage burn be α then at the end of the first stage burn the rocket mass will be represented by M_1

$$M_1 = m_{struct} + (1-\alpha)m_{fuel} + m_{sat} \qquad 2$$

Tsiolkovski Formula

$$v_2 - v_1 = v_p.Log_e \frac{m_1}{m_2} \qquad \Delta v = v_p.LN.\frac{m_1}{m_2} \qquad 3$$

Applying Tsiolkovski's formula to the first stage we get a mass ratio of :

$$\left[\frac{M_L}{M_1}\right]_{First} = \frac{m_{struct} + m_{fuel} + m_{sat}}{m_{struct} + (1-\alpha)m_{fuel} + m_{sat}} \qquad 4$$

Where
α is the proportion of fuel mass used in the first stage
and the structure mass is assumed to be proportional to fuel mass required.

For the second stage we assume that the structure mass of the first stage has been removed following seperation

$M_2 = (1-\alpha)\, m_{struct} + (1-\alpha) m_{fuel} + m_{sat}$ 5

$M_3 = (1-\alpha)\, m_{struct} + m_{sat}$ 6

So applying Tsiolkovski's formula to the second stage we get a mass ratio of :

$$\left[\frac{M_2}{M_3}\right]_{Second} = \frac{(1-\alpha)m_{struct} + (1-\alpha)m_{fuel} + m_{sat}}{(1-\alpha)m_{struct} + m_{sat}} \quad 7$$

Tsiolkovski Formula

$$v_2 - v_1 = v_p . Log_e \frac{m_1}{m_2}$$

Now introducing the mass ratios of the two stages in the Tsiolkovski formula we have :

$$\Delta v_{1+2} = v_p . \left[LN\left(\frac{M_L}{M1}\right)_{First} + LN\left(\frac{M_2}{M_3}\right)_{Second} \right] \quad 8$$

For a single stage

$$\Delta v = v_p . LN\left(\frac{M_0}{M_1}\right) \quad 9$$

So the multiplying factor for using a two stage rocket launching system is given by the following

$$\frac{\Delta v_{1+2}}{\Delta v} = v_p . \left[\frac{LN\left(\frac{M_L}{M_1}\right) + LN\left(\frac{M_2}{M_3}\right)}{LN\left(\frac{M_0}{M_1}\right)} \right] \quad 10$$

Excel table with worked example